B.Sc-Part-II(H), Paper-III Curvature.

28-07-2020

Q.1. Find the radius curvature for the polar curve $r = f(\theta)$

or, Prove that the formula $\rho = \dfrac{(r^2+r_1^2)^{\frac{3}{2}}}{r^2+2r_1^2-rr_2}$, where the symbols have their ~~what~~ usual meaning.

Soln: We know that $\dfrac{1}{p^2} = \dfrac{1}{r^2} + \dfrac{1}{r^4}\left(\dfrac{dr}{d\theta}\right)^2$ ——— (1)

Differentiating both sides with respect to θ, we have

$$-\frac{2}{p^3}\frac{dp}{d\theta} = -\frac{2}{r^3}\cdot\frac{dr}{d\theta} - \frac{4}{r^5}\left(\frac{dr}{d\theta}\right)^3 + \frac{2}{r^4}\frac{dr}{d\theta}\cdot\frac{d^2r}{d\theta^2}$$

$$\text{or, } \frac{dp}{dr} = \left\{\frac{1}{r^3} + \frac{2}{r^5}\left(\frac{dr}{d\theta}\right)^2 - \frac{1}{r^4}\frac{d^2r}{d\theta^2}\right\}p^3$$

$$= \left\{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}\right\}\cdot\frac{p^3}{r^5} \quad \text{——— (2)}$$

From (1) $\dfrac{1}{p^2} = \dfrac{r^2 + \left(\frac{dr}{d\theta}\right)^2}{r^4}$

$$\text{or, } p = \frac{r^2}{\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}} \quad \text{——— (3)}$$

We know that $\rho = r\dfrac{dr}{dp}$ [by (2)]

$$= r\,\frac{r^5}{p^3\left\{r^2+2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}\right\}}$$

$$= \frac{\cancel{r^6}\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}^{\frac{3}{2}}}{\cancel{r^6}\left\{r^2+2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}\right\}}$$

Hence, $\rho = \dfrac{(r^2+r_1^2)^{\frac{3}{2}}}{r^2+2r_1^2-rr_2}$ Proved.

Q.2. Find the radius of curvature when x and y are known functions of s. 28-07-2020

or, Prove that the formula: $\frac{1}{\rho^2} = \left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2$.

Soln: We know that

$$\cos\psi = \frac{dx}{ds} \text{ and } \sin\psi = \frac{dy}{ds}$$

Differentiating with respect to s, we have

$$-\sin\psi \frac{d\psi}{ds} = \frac{d^2x}{ds^2} \quad \text{—— (I)}$$

and $$\cos\psi \frac{d\psi}{ds} = \frac{d^2y}{ds^2} \quad \text{—— (II)}$$

Squaring and adding (I) and (II), we get

$$\left(\frac{d\psi}{ds}\right)^2 (\sin^2\psi + \cos^2\psi) = \left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2$$

Hence, $$\frac{1}{\rho^2} = \left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2, \text{ where } \rho = \frac{ds}{d\psi}$$

Q.3. Prove that $r^2 = p^2 + \left(\frac{dp}{d\psi}\right)^2$

and $\rho = p + \frac{d^2p}{d\psi^2}$

Soln: We know that $p = r\sin\phi$ —— (1)

$\cos\phi = \frac{dr}{ds}$ —— (2)

and $\rho = r\frac{dr}{dp}$ —— (3)

Now, $\frac{dp}{d\psi} = \frac{dp}{dr}\cdot\frac{dr}{ds}\cdot\frac{ds}{d\psi} = \frac{dp}{dr}\cos\phi\cdot\rho$ $\left[\because \rho = \frac{ds}{d\psi}\right]$

$= \frac{dp}{dr}\cos\phi \cdot r\frac{dr}{dp}$ from (3)

$= r\cos\phi$ —— (4)

Squaring and adding (1) and (4), we obtain

$$p^2 + \left(\frac{dp}{d\psi}\right)^2 = r^2(\sin^2\phi + \cos^2\phi) = r^2$$

Diff. w.r.to p, we get

$$2p + 2\frac{dp}{d\psi}\cdot\frac{d^2p}{d\psi^2}\cdot\frac{d\psi}{dp} = 2r\frac{dr}{dp}$$

i.e. $p + \frac{d^2p}{d\psi^2} = r\frac{dr}{dp} = \rho$

Hence, $\rho = p + \frac{d^2p}{d\psi^2}$ Proved